Mathematical Analysis of Top-Ranked Programs

The following section represents the work of Drs. King and Bright in reviewing the mathematical soundness of the top four high school curricular materials for Algebra 1 and 2, Geometry and Integrated Mathematics 1, 2 and 3. The team selected key standards that represent important development of mathematical concepts that allow students to be well-prepared to continue in mathematics study. The selection of these standards does not imply that these are more valuable than others; it simply provided a method for deep analysis on central themes.

Review of Mathematical Soundness of High School Curriculum Materials
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The OSPI alignment study of high school curriculum materials was organized in three categories: Algebra 1/Algebra 2 materials, Geometry materials, and Integrated Mathematics materials. This review of mathematical soundness is organized in the same way. For each category, the Performance Expectations that drove the review are listed first. However, we did not replicate the alignment study that OSPI has already completed. Rather, we looked for evidence of mathematical soundness; that is, mathematical correctness and coherent development of ideas. Only the best-aligned materials (based on preliminary analysis of the OSPI alignment study) were reviewed; the order of these reviews reflects the order of these materials in the preliminary data analysis. A summary/synthesis of the reviews is provided at the end of each section. Any review of mathematical soundness of necessity reflects reviewers' views about mathematics itself about how an idea is, or should be, explained. Different mathematicians will potentially have different views on the "best way" to present an idea so that it is clear. Geometers and topologists, for example, "see" mathematical ideas differently, even though they study some of the same mathematical objects. No review is likely to represent all possible views. We were looking for evidence that materials provided opportunities for students to develop mathematical understanding that would be rich and deep, as opposed to compartmentalized. In general, the materials we reviewed were found to be mathematically sound. However, we found differences among the materials related to the development of rich, deep mathematical understanding. These differences might be important to districts as they consider choosing materials for instructional use.

4.1 Algebra 1/Algebra 2

One of the major organizing ideas in algebra is *functions*. Students in Algebra 1/Algebra 2 are expected to become very familiar with linear, quadratic, and exponential functions and to gain some experience with other kinds of functions. There are many ways that the mathematics ideas related to functions might be examined. We have chosen two categories of ideas.

First, we chose to examine the development of one class of functions. The class of functions that seems most extensively developed in the high school PEs is quadratics; this is an important class of functions for high school students, both for developing mathematical maturity and in terms of application to science.

The relevant PEs are listed below.
A1.1.D (M2.1.B) Solve problems that can be represented by quadratic functions and equations.
A1.5.A (M2.2.A) Represent a quadratic function with a symbolic expression, as a graph, in a table, and with a description, and make connections among the

representations.
A1.5.B (M2.2.B) Sketch the graph of a quadratic function, describe the effects that changes in the parameters have on the graph, and interpret the x-intercepts as solutions to a quadratic equation.
A1.5.C (M2.2.D) Solve quadratic equations that can be factored as (ax + b)(cx + d) where a, b, c, and d are integers.
A1.5.D (M2.2.F) Solve quadratic equations that have real roots by completing the square and by using the quadratic formula.
A2.3.A (M2.2.C) Translate between the standard form of a quadratic function, the vertex form, and the factored form; graph and interpret the meaning of each form.
A2.3.B (M2.2.E) Determine the number and nature of the roots of a quadratic function.
A2.3.C (M2.2.G) Solve quadratic equations and inequalities, including equations with complex roots.

To a lesser extent, we also examined how some general ideas related to function were developed. Understanding domain/range, developing skill at moving among representations of functions, and identifying the role that parameters play are all important ideas. The Performance Expectations (PEs) below provide focus for these ideas.

A1.3.A (M1.2.A) Determine whether a relationship is a function and identify the domain, range, roots, and independent and dependent variables.
A1.3.B (M1.2.B) Represent a function with a symbolic expression, as a graph, in a table, and using words, and make connections among these representations.
A1.4.E (M1.3.B) Describe how changes in the parameters of linear functions and functions containing an absolute value of a linear expression affect their graphs and the relationships they represent.
A1.5.B (M2.2.B) Sketch the graph of a quadratic function, describe the effects that changes in the parameters have on the graph, and interpret the x-intercepts as solutions to a quadratic equation.
A1.7.A (M1.7.A) Sketch the graph for an exponential function of the form $y = ab^n$ *where n is an integer, describe the effects that changes in the parameters a and b have on the graph, and answer questions that arise in situations modeled by exponential functions.*

4.1.1 Discovering Algebra/Discovering Advanced Algebra

In *Discovering Algebra*, significant groundwork for the study of functions is laid in Chapter 7. It is significant that the ideas are developed here for functions in general; this creates a coherent mathematical sequence that is critical for helping students “see” the mathematical big picture. Domain and range for relations and functions are introduced in Lesson 7.1 and reinforced throughout the chapter. The vertical line test is introduced in Lesson 7.2, with application to the graphs of a wide range of functions/relations. Lessons 7.3 and 7.4 develop critical understanding of how functions can be used to represent different contexts; this helps motivate the need to study special kinds of functions, beginning in Lesson 7.5 (absolute value function) and Lesson 7.6 (parabolas).

Chapter 8 (Transformations of Functions) provides general background on how different function rules (e.g., $y = |x|$ and $y = |x| + 3$ or $y = x^2$ and $y = x^2 + 3$) generate graphs that look the same but are in different positions through translation, reflection, and scaling. Dealing with these issues in general prevents the need to deal with a collection of special cases when quadratic functions are studied (Chapter 9). This approach provides coherence to the mathematics ideas and would seem to make the mathematics more easily learned. For example, when students encounter Chapter 9, they will already know the effect of changing the value of *a* in the equation, $y = ax + b$.

Chapter 9 deals with quadratic functions. The introduction is through the modeling of real-world situations, but more standard ideas are addressed almost immediately: roots and vertex (Lesson 9.2), vertex and general form (Lesson 9.3), factoring (Lesson 9.4), completing the square (Lesson 9.6), and quadratic formula (Lesson 9.7). The extension to cubic equations (Lesson 9.8) provides a "non-example" that helps cement understanding of properties of quadratic functions. The development of critical ideas earlier in the context of many different functions should help students develop rich cognitive understanding that can be retained permanently.

In *Discovering Advanced Algebra* functions and transformations of functions are addressed in Chapter 4; again, the ideas are applied to a range of functions as a means of illustrating the power of these ideas. Lesson 4.4 specifically addresses transformations of quadratic functions. Chapter 7 (Quadratic and Other Polynomial Functions) provides specific review and extension of the study of quadratic functions. Topics include finite differences (Lesson 7.1), equivalent forms/rules (Lesson 7.2), completing the square (Lesson 7.3), quadratic formula (Lesson 7.4), and complex numbers (Lesson 7.5) which allows factoring of previously "unfactorable" quadratic expressions. Extension to higherorder polynomials provides a contrast quadratic functions; having examples and nonexamples of the relevant ideas is important for helping students generalize accurately.

In general, the "Discovering" series strikes a very good balance between teaching general concepts/skills (e.g., transformations of functions) and specific concepts/skills related to quadratic functions (e.g., equation of the line of symmetry of a parabola). The mathematics is developed coherently (and soundly). By the end of the Advanced Algebra course, students should be quite ready to move on to pre-calculus.

4.1.2 Holt Algebra 1/Algebra 2

In *Algebra 1,* functions as rules are introduced in Chapter 1, but the ideas are not developed until Chapter 4. Operations on polynomials, factoring, and quadratic functions are addressed in Chapters 7, 8, and 9.

In Chapter 4, graphs are used to represent situations. Then the standard characteristics of functions are discussed: relations and functions (Lesson 4-2), vertical line test (Lab Lesson 4-2), function rules (Lesson 4-3), graphing (Lesson 4-4), and multiple representations of functions (Technology Lab Lesson 4-4). These ideas are treated somewhat compartmentally, however.

The second half of Chapter 7 addresses addition, subtraction, and multiplication of polynomials, including special products of binomials (i.e., squares of binomials and product of sum and difference of two quantities). Algebra tiles are used to model the ideas, but symbolic manipulation (including FOIL) is the technique used in the worked out examples in the lessons.

Chapter 8 addresses factoring, first for monomials and then of general trinomials (i.e., $x^2 + bx + c$ and $ax^2 + bx + c$), with special products (e.g., difference of two squares) following. In worked-out examples, factoring is completed by identifying combinations of the factors of *c* and *a* to generate *b*. The modeling with algebra tiles in the introductory Lab Lesson is not extended into the "regular" lessons. Lesson 8-6 brings all of the techniques together by discussing "choosing a factoring method;" this is a nice way to help students reflect on what they have learned in the chapter.

Chapter 9 deals with quadratic functions. In Lesson 9-1 the idea of constant second differences is introduced and related to constant first differences already developed for linear functions. Lab Lesson 9-2 provides an opportunity for explorations leading to the equation for the axis of symmetry. Additional worked-out examples highlight relationships among the zeros, the axis of symmetry, and the vertex; graphing of parabolas (Lesson 9-3) is centered around these relationships. Families of quadratic functions (Lab Lesson 9-4) and transformations (Lesson 9-4) build on the ideas developed about graphing. The second half of the chapter deals with solving quadratic equations, completing the square, and the quadratic formula.

In *Algebra 2* functions are reviewed and extended in Chapter 1; this includes attention to transformations of functions and an emphasis on "parent" functions. Chapter 5 (Quadratic Functions) begins from this orientation of parent functions and leads to the vertex form of the quadratic equation. This is a very nice way to provide conceptual grounding for the entire chapter. Lab Lesson 5-3 connects the graph of a quadratic and the graphs of the factors of the quadratic expression; this, too, provides very good conceptual underpinning for understanding characteristics of quadratic functions. The primary extension for the remainder of this chapter is complex numbers, with applications to solving quadratic equations with no real roots.

Although the sequence of ideas in this series is fairly traditional, opportunity is provided for students to make connections among the ideas. It seems likely that students will exit with a rich understanding of the mathematics ideas underlying quadratic functions. Mathematical soundness, thus, is clearly evident.

4.1.3 Glencoe/McGraw Hill Algebra 1/Algebra 2

Relations and functions are introduced in Chapter 1, but quadratic functions are not addressed directly until Chapters 7-9. The time lag (Chapters 2-6 deal with linear equations, functions and inequalities.) might make it necessary essentially to re-teach the generic ideas at that time.

Chapter 7 deals with operations on polynomials. This is mainly a skills chapter; the word problems included seem somewhat forced. There are many exercises in each lesson (e.g., 89 exercises for lesson 7-2); it is not clear why so many similar exercises are needed. The use of algebra tiles to model operations is very nice; this sets the stage for use of this representation in Chapter 8 for factoring of trinomials. This model is explicitly tied to both horizontal and vertical symbolic recording processes for the operations on polynomials. One concern here is that students will not have much motivation to learn the skills, so they may try to memorize (rather than learn) the skills. The sequencing of the lessons and the presentation of the mathematics would seem to encourage this approach. Providing a rationale for learning this material would be a welcome addition.

Chapter 8 deals with factoring and solving quadratic equations. Again, this material is approached mainly as a sequence of skills, rather than with some underlying conceptual underpinning. Ideas addressed include factoring monomials (Lesson 8.1), factoring using the distributive property (Lesson 8.2), and factoring trinomials (Lesson 8.3). It is important that general trinomials (i.e., $ax^2 + bx + c$) are addressed first, initially through the model provided by algebra tiles. Differences of square and perfect squares are presented as special cases of the general case. This seems to be a good approach, since it puts the emphasis correctly on general ideas.

Chapter 9 deals with quadratic and exponential functions, though more emphasis is given to quadratic functions here. Lesson 9-1 introduces graphs of quadratic functions and simply states "facts" about quadratic functions (e.g., the axis of symmetry is $x = -(b/2a)$), without providing a clear rationale for why these facts are true. This approach would seem to encourage students to memorize information rather than trying to understand that information. Subsequent topics include solving by graphing (Lesson 9-2), transformations (lesson 9-3), completing the square (Lesson 9-4), and quadratic formula (Lesson 9-5). Lessons 9-6 through 9-9 provide experience with exponential functions and finite differences. As in earlier chapters, there are many exercises (e.g., 95 for Lesson 9-1), without any obvious reason for so many.

The sequencing of ideas in this Algebra 1 book is quite traditional. There seems to be an over-emphasis on skill development rather than conceptual development. However, this approach lends itself to a relatively close alignment of the book to almost any set of standards. The sequence of lessons would be understandable to most high school mathematics teachers, even though it might not generate a coherent "view" of mathematics ideas among novices (i.e., students).

Algebra 2 addresses quadratic functions mainly in Chapter 5. The work from *Algebra 1* is revisited, with extensions of some work to complex numbers. In this course, too, some key facts (e.g., "A quadratic equation can have one, two, or no real solutions." p. 260) are simply stated, without any rationale, other than examples, for why those facts are true. If teachers do not emphasize the examples adequately, this approach would seem to encourage memorization. The development of transformations of quadratic functions is done more completely here than in the earlier book.

Chapter 6 addresses operations (including division) on polynomials, and polynomial functions. This work goes beyond that required by the Algebra 2 Standards, but it is organized to help students gain insight into an important set of mathematical ideas (e.g, rational zero theorem). This seems to be a nice extension of work with quadratic functions. Lesson 10-2 also deals with parabolas as part of the study of conic sections.

Overall, the mathematics is sound, though there is probably not enough rationale provided for helping students *want* to learn the mathematics. The approach is heavily oriented toward skill development.

4.1.4 Prentice Hall Algebra1/Algebra 2

In *Algebra 1* the concept of function is introduced in chapter 1, along with domain and range. This lays general background for later work, even though there is not much development here.

Functions reappear in much more depth in Chapter 5, which is a general discussion of functions. First, functions are used as models for events (Lesson 5-1). This is followed by relations and functions (Lesson 5-2), rules, tables, and graphs (Lesson 5-3), and four lessons on writing and using function rules. These four lessons seem to present the mathematics as compartmentalized ideas, somewhat disjoint from each other. There is no apparent underlying common thread that ties the ideas together.

Chapter 9 is focused on operations on polynomials and factoring. Algebra tiles are used as a model for multiplication of binomials, with connections made to both vertical and horizontal recording schemes. Factoring is introduced first for $x^2 + bx + c$ (i.e., finding factors of C whose sum is b; Lesson 9-5) and then $ax^2 + bx + c$ (i.e., "reverse application of FOIL"; Lesson 9-6). Special cases of difference of two squares and perfect squares (Lesson 9-7) are presented through rules as well as examples. Algebra tiles are used in an activity lab, but do not appear as part of the primary focus on instruction.

Chapter 10 begins with graphing of special cases of quadratic functions (Lessons 10.1), namely, $y = ax^2$ and $y = ax^2 + c$. Then the general case is presented (Lessons 10.2), along with graphing of inequalities. It is not clear why the special cases need to be presented first. There is a short demonstration that attempts to justify the equation of the axis symmetry. In Lesson 10-3 quadratic equations are solved by graphing, along with use of square roots to solve $ax^2 + c$, but these strategies are not connected in any way. Lesson 10-4 is factoring to solve quadratic equations, followed by completing the square (Lesson 10-5), quadratic formula (Lesson 10-6), discriminant (Lesson 10-7), and modeling (Lesson 10-8). Instruction is through worked-out examples followed by exercises. The mathematics is correct, and the sequence would probably be comfortable to most high school mathematics teachers, but there is very little help provided for students in understanding how these ideas and skills tie together. Ideas are presented in a compartmentalized way.

In *Algebra 2*, the work is reviewed and extended. There is still a tendency to reduce ideas to a series of "cases." For example, Lesson 5-4 on factoring has worked-out examples for several cases: (1) $ac > 0$ and $b > 0$, (2) $ac > 0$ and $b < 0$, (3) $ac < 0$, (4) $a \neq 1$ and $ac > 0$, and (5) $a \neq 1$ and $ac < 0$. This could clearly create the impression that identifying what case "applies" is the first step in determining how to factor a trinomial, followed by applying some memorized procedures for that case. This makes the issue of factoring an overwhelming learning burden. The major extension in this chapter is work with complex numbers, so that completing the square and quadratic formula work can include imaginary solutions.

Overall, the mathematics is sound, though there is not enough rationale provided for helping students *want* to learn the mathematics. The sequencing of examples and procedures tends to create an impression that there are many distinct "cases" that students should remember. There is too little attempt to "combine" cases under some general umbrella so that students understand how the cases are related to each other.

4.1.5 Conclusions: Algebra 1/Algebra 2

All four series provide coverage of mathematically sound content. The Discovering series and the Holt series seem to be the ones that tie together key mathematics ideas best. Since coherence of mathematics ideas is a part of mathematical soundness, these two series rate high. The Glencoe and Prentice Hall series leave an impression of compartmentalization of ideas. These two series rate somewhat lower, though they are still mathematically sound. Teachers might have to work harder to ensure that students develop deep understanding.

4.2 Geometry

One of the major themes in the Geometry standards is proof. It is clearly important to develop the idea of proof rigorously. One other major theme in Geometry is continued development of properties of figures. We have chosen to focus on parallel/perpendicular lines and parallelograms. The relevant Performance Expectations are listed below.

G.1.A (M1.4.A) Distinguish between inductive and deductive reasoning.

G.1.B (M1.4.B) Use inductive reasoning to make conjectures, to test the plausibility of a geometric statement, and to help find a counterexample.

G.1.C (M1.4.C and M2.3.A) Use deductive reasoning to prove that a valid geometric statement is true.

G.1.D (M2.3.C) Write the converse, inverse, and contrapositive of a valid proposition and determine their validity.
G.1.E (M2.3.B) Identify errors or gaps in a mathematical argument and develop counterexamples to refute invalid statements about geometric relationships.
G.1.F (M2.3.D) Distinguish between definitions and undefined geometric terms and explain the role of definitions, undefined terms, postulates (axioms), and theorems.
G.2.A (M1.4.E) Know, prove, and apply theorems about parallel and perpendicular lines.
G.2.B (M1.4.F) Know, prove, and apply theorems about angles, including angles that arise from parallel lines intersected by a transversal.
G.2.C (M1.4.G) Explain and perform basic compass and straightedge constructions related to parallel and perpendicular lines.
G.3.F (M2.3.J) Know, prove, and apply basic theorems about parallelograms.
G.3.G (M2.3.K) Know, prove, and apply theorems about properties of quadrilaterals and other polygons.
G.4.A (M1.3.H) Determine the equation of a line in the coordinate plane that is described geometrically, including a line through two given points, a line through a given point parallel to a given line, and a line through a given point perpendicular to a given line.
G.4.B (M2.3.L) Determine the coordinates of a point that is described geometrically.
G.4.C (M2.3.M) Verify and apply properties of triangles and quadrilaterals in the coordinate plane.

What is called for is a set of theorems stating properties of parallelograms. What is needed for this are the basic theorems about angles formed by parallels and a transversal, along with the angle sum theorem for polygons and some congruence theorems for triangles. In the reviews that follow, these topics will be referred to as the standard parallelogram theorems.

4.2.1 Holt Geometry

Chapter 2 contains an extensive development of inductive and deductive reasoning, including formal rules of logic. Section 2.1 introduces inductive reasoning and conjecturing in mathematics, science, and life outside science. Next come Venn diagrams and Section 2.2 on conditional (if-then) statements. Section 2.3 addresses deductive reasoning as a way to verify conjectures. Section 2.4 is devoted to bi-conditional statements and definitions. Section 2.5 addresses algebraic proof, and Sections 2.6 and 2.7 begin geometric proof - two-column and then flowchart and paragraph proofs. All sections include a generous selection of examples and problems from geometry, other areas of mathematics, and daily life. Various strategies and representations are presented to support understanding and applications of these ideas. These rules of logic and proof are used to develop geometry topics in the rest of the book.

Chapter 3 focuses on parallel and perpendicular lines. Section 3.1 provides definitions of parallel and perpendicular lines, as well as skew lines and parallel planes. This is followed by an informal introduction to examples of parallel lines (e.g., the edges of a box). Terminology is developed here for the four pairs of angles formed by two lines and a transversal line. Section 3.2 begins with a postulate (Postulate 3-2-1) that states the equality of corresponding angles in a figure formed of two parallel lines and a transversal. Then the consequences are stated and proved as examples or problems. Section 3.3 includes a new postulate (Postulate 3-3-1) that is the converse of Postulate 3-2-1; that is, sufficient conditions that two lines be parallel. This postulate is used to prove theorems establishing that certain lines are parallel, including the case of two lines perpendicular to the same line. Section 3.3 ends with a Geometry Lab with constructions for parallel lines by compass and straightedge and by paper folding.

Section 3.4 is devoted to perpendicular lines, including some theorems about perpendicular transversals and compass and straightedge construction of the perpendicular bisector of a segment. There it is also a statement that the shortest segment from a point to a line is the perpendicular segment (the proof will come later). The Geometry Lab introduces constructions of perpendicular lines. Sections 3.5 and 3.6 deal with lines in the coordinate plane. Intersections of lines are found by solving linear equations; the concept of slope is developed and it is asserted as a theorem that parallel lines have the same slope and that perpendicular lines have slopes whose product is -1. The relationships between slope and parallelism are neither proved nor justified informally.

This chapter does a thorough job of stating and proving the basic angle theorems about parallel lines and transversals and also theorems about perpendicular lines. The inclusion of some properties of distance in the section on perpendiculars seems natural, though it does require assuming a theorem whose proof must be deferred. It is puzzling that there is no attempt to explain the slope relations for parallel and perpendicular lines, either by solving simultaneous algebraic equations or drawing simple figures with slope. This is a missed opportunity to help students make sense of the mathematics.

Chapter 6 (Parallelograms and Polygons) begins by introducing some basic definitions and theorems about polygons in general and developing the theory of parallelograms. A later part of the chapter moves on to special parallelograms and other special quadrilaterals such as isosceles trapezoids and kites. Section 6.1 defines basic terminology such as vertex, interior angle, exterior angle, and then states and proves theorems for general convex n-gons about the sum of the interior angles and the sum of the exterior angles (an important theorem that is not always given the prominence that is its due). Section 6.2 develops the standard properties of parallelograms. The properties are proved as theorems and also are studied by construction and drawing, and there are examples in the coordinate plane. Section 6.3 proves and applies conditions for parallelograms, that is, the converses of some of the theorems of 6.2. Examples and problems in the coordinate plane apply some of these theorems. Section 6.4 is about properties special parallelograms. These include parallelograms with adjacent angles equal (rectangles) and those with adjacent sides equal (rhombi). It is pointed out that squares are parallelograms with both properties. Section 6.5 proves and applies conditions for special parallelograms, including examples in the coordinate plane. The remaining sections of this chapter are devoted to other special quadrilaterals such as isosceles trapezoids and kites.

Sections 6.2 and 6.3 and Sections 6.4 and 6.5 follow a pattern of paired sections found often in this text. Certain proofs are given in the first section of the pair and then converses are developed in second section. Throughout the chapter, there are mathematically illuminating applications of parallelograms and special quadrilaterals, from carpentry to mechanical devices (e.g., car jacks).

In Summary, *Holt Geometry* includes a full treatment of what is required by the Standards and a bit more. The mathematics is developed rigorously, with proofs of theorems based on postulates. Many of the examples and exercises are either proofs of these theorems or applications of them to geometry problems. In addition there are examples of applications and some geometry lab experiments with constructions.

4.2.2 McDougal-Littell Geometry

Chapter 2 (Reasoning and Proof) begins with an extensive Section 2.1 explicitly on inductive reasoning. This features numerical and geometrical patterns and examples about data. Section 2.2 addresses conditional statements, including if-then statements and their converses, contrapositives and inverses, and the relationship between definitions and biconditional statements. Some examples address perpendicular lines and vertical angles. This section is rather short in exposition, but there are several pages of exercises. Section 2.3 is about applications of deductive reasoning, including statements of the Law of Detachment and the Law of Syllogism. Examples involve mathematics and the real world, but not much about geometry is proved in this section. An extension addresses symbolic notation, including the standard arrow notation and truth tables. Section 2.4 includes a list of postulates about the incidence relations among points, lines, and planes along with some interesting comments about how to interpret geometrical diagrams and what can be assumed in diagrams. Solution of algebraic equations is reviewed in Section 2.5. Section 2.6 (Prove Statements about Segments and Angles) includes proofs of minor results about lengths of segments and measure of angles. An example of how to write a two-column proof is provided in one example. Section 2.7 establishes standard angle pair relationships, including the congruence of right angles and the vertical angle theorem. Overall, this chapter presents the rules of logic and proof. However, the examples and illustrations seem not to go very far in addressing the difficulties inherent in understanding these concepts. The examples of proofs are technical and minor, with little geometric interest.

Section 3.1 (Identify Pairs of Lines and Angles) begins with postulates that state for a given line and a point, there is exactly one line through the point parallel to the line and one perpendicular to the line. The usual terminology is defined for pairs of angles formed by two lines and a transversal, but no theorems are proved in this section. In Section 3.2, a Corresponding Angle Postulate is stated (even though this is a really a theorem that follows from the parallel postulate in 3.1). Then three additional congruence theorems (one example and two exercises) are proved about pairs of angles defined by two parallels and a transversal. In Section 3.3 the converses of the theorems from 3.2 are proved (sufficient conditions for lines to be parallel). These theorems are used to prove the important fact that the parallel relation is transitive. Most of the exercises are immediate applications of the theorems. Sections 3.4 and 3.5 are about equations of lines. In 3.4, slope is defined and there are postulates that state if-and-only-if conditions on the slope for lines to be parallel or perpendicular. There is no indication that these properties can in fact be proved and do not need to be assumed as postulates. Section 3.6 is devoted to proving theorems about perpendicular lines. There is a proof that a linear pair of congruent angles is a pair of right angles and relates this to the real-world consequence of folding paper. Special cases of parallels and transversals when the transversal is perpendicular are spelled out. One strong feature of Chapter 3 is the explicit attention to the transitive property of parallelism. One weakness is the redundancy of assuming a parallel postulate and then assuming an equivalent statement as a postulate in the next section rather than proving it as a theorem (or at least noting that it can be done). Another weakness is the absence of any explanation or proof for the slope properties of parallels and perpendiculars, or even noting that these properties are really theorems, not postulates.

The topic of parallelograms appears rather late (Chapter 8), after a chapter on right angle trigonometry. Section 8.1 states the interior and exterior angle sum theorems for convex polygons (proofs are exercises). This is a short section with a few examples and exercises. The problem of finding the angle sum of a convex polygon is presented as a challenge but the figures supplied as hints and the answer key are incomplete in that they assume the polygon can be dissected into triangles, all of which have the same shared vertex. This teacher notes do not alert the teacher to the underlying mathematical difficulty, so the opportunity for a more challenging discussion is not supported. In Section 8.2 the usual properties of a parallelogram are stated and proved in exercises. In some problems in the coordinate plane, students are simply told that quadrilaterals are parallelograms, when students could (and should) verify this fact. Section 8.3 states the four standard necessary criteria for a quadrilateral to be a parallelogram; the opposite sides congruent theorem is proved as an example and the others are left to exercises. Here, there is a demonstration that a quadrilateral in the coordinate plane is a parallelogram by showing that one pair of sides is congruent and parallel. Students are asked to use other methods to verify that the quadrilateral is a parallelogram. An appendix to Section 8.3 is a Problem Solving Workshop that demonstrates two methods for determining whether or not a figure in the coordinate plane is a parallelogram. This is a valuable addition to the section. Section 8.4 contains if-and-only-if conditions for quadrilaterals to be rhombuses, rectangles, and squares. A Venn diagram shows how the set of squares is the intersection of the set of rhombuses and the set of rectangles. A definition of a square is given here, but rectangles and squares have been used regularly in earlier chapters (e.g., in the proofs of the Pythagorean theorem). There is no acknowledgement of the earlier appearance of squares when squares are defined in this chapter. This undercuts the presentation of geometry as an axiomatic and logical system.

The *McDougal-Littell* text covers the Washington Standards items checked in this review, but the impression of the mathematics in this text is mixed. The reasoning section seems rather shallow, though there is good discussion about how to reason from figures. The exercises routinely have examples of incorrect proofs in which students are asked to find the error. There is more attention than usual devoted to the transitive property of parallelism, and there is an extra section with explicit examples of multiple solutions of a problem. On the other hand, most of the exercises are routine or else do not really exploit the mathematical possibilities of potentially rich problems. Whether or not it is a good choice to postpone parallelograms and rectangles to the second half of the text is something that should be considered. Rectangles and squares appear informally in many earlier places in the text without any explicit efforts to reconcile the delay of rigorous development. Teachers will have to deal with possible confusion coming from this departure from logical development.

4.2.3 Glencoe McGraw-Hill Geometry

Chapter 2 addressed reasoning and proof. Section 2.1 presents inductive reasoning as using examples to form a conclusion that may – as a conjecture – lead to a prediction. Several contexts are presented, including number sequences, geometrical figures, and data. Section 2.2 introduces some aspects of formal logic including truth tables, conjunctions, and disjunctions. (The book uses this technical terminology for logical "and" and "or.") Venn diagrams are also introduced. Section 2.3 is about conditional (ifthen) statements; mathematical and real world examples are included. The converse, inverse, and contrapositive are defined, and there is a proof using truth tables showing which statements are equivalent. There is an extension about bi-conditional statements. Section 2.4 introduces deductive reasoning, including the Law of Detachment and the Law of Syllogism. An extensive set of examples is given, some of which are quite illuminating about the uses of if-then statements and possible pitfalls in understanding them. A data analysis example used to provide a contrasting example with inductive reasoning. Section 2.5 is about postulates and paragraph proofs. Some postulates about the relations among points, lines and planes are presented and then some proofs are based on these postulates. This is all correct, but the modest toolkit of postulates at this point limits the interest and challenge of what can be proved. The chapter concludes with Sections 2.6 (algebraic proof), 2.7 (proving segment relationships), and 2.8 (proving angle relationships). These sections focus on short proofs of technical and rather trivial propositions. This writing in this chapter is not a clear development of the mathematical ideas. Some helpful examples are included, but others range so far afield that they are a distraction from what is important for proof in geometry. The chapter may unintentionally communicate that the goal of proof is to find the right terminology rather than to find reasons for important mathematical statements. This seems to divert attention away from the study of geometry. In writing mathematics logically, more technical detail is not necessarily better. Focus on, and clarity about, the mathematics content being studied is essential.

Chapter 3 is devoted to parallel and perpendicular lines. Section 3.1 defines parallel and skew lines, as well as parallel planes, with exercises to find such lines in a wedge of cheese or a cubical box. Terminology about angle pairs defined by a transversal is introduced, along with practice using this terminology. Section 3.2 is about angles and parallel lines. Based on a postulate about corresponding angles, the congruence of other angle pairs is proved. The special case of a perpendicular transversal is a theorem, and there are examples and exercises about angle measures in geometry figures and in realworld examples. Section 3.3 includes postulates about the slope relationships for parallel and perpendicular lines; there are no explanations for why these are true. In Section 3.4, most of the work is finding the equations of lines through two points, but there is also an example of a line through a point that is parallel to a given line. In the Geometry Lab at the end there is a more substantial example developed, which is to find the equation of the perpendicular bisector of a segment in the coordinate plane.

Section 3.5 is about proving lines are parallel in the plane. Postulate 3.4 asserts that if two lines are cut by a transversal so that all the corresponding angles are congruent, then the lines are parallel. This is followed by a description of the construction of a line through a point parallel to a given line. Then comes Postulate 3.5, which is a version of the Euclidean parallel postulate. Next are four theorems that state the congruence of a pair of angles implies that two lines are parallel. The proofs are left to the exercises. Several aspects of the mathematical development in this section are troubling. First, Postulate 3.4 is unusual and awkward, since it is sufficient that only one pair of the corresponding angles be congruent. In fact the statement that one pair of corresponding angles is missing, though one theorem correctly asserts that if one pair of congruent alternating interior angles implies the lines are parallel. Second, there is the curious appearance of the Euclidean Parallel Postulate. It is stated that the straightedge and compass construction proves that there is at least one parallel line, but this Postulate is needed to prove that there is only one. However, the two postulates about corresponding angles already given are sufficient to prove the Euclidean Parallel Postulate, so the insertion of this additional postulate is unnecessary and confusing. Also, the historical note (i.e., Euclid needed only five postulates to prove the theorems “in his day”) is very odd.

Section 3.6 on perpendiculars and distance begins by asserting without proof that the distance from a point to a line is the length of the perpendicular segment from the point to the line. The uniqueness of the perpendicular is stated as a Postulate in the text, but the fact that the length is minimal is not justified. At the end of Section 3.6, the concept of distance between two parallel lines is introduced as the distance from any point on one line to the other line. This is followed by a detailed example in which the distance between two parallel lines in the coordinate plane is computed. This section has some logical difficulties. Early on, an alternate definition of parallel lines is given; namely, two lines are parallel if they are equidistant. Since the proof of equidistance depends on rectangle properties that are not yet developed, the definition can only be stated here without proof. If distance is going to enter into this chapter, there should at least be a coherent explanation so that it is clear that there are statements that must be proved later, so that students will not be confused about the underlying mathematics. Worse, students are asked to prove that if two lines are equidistant from a third, then the two lines are parallel. Since the logical development is deficient here, no proof could be correct.

The answer in the teacher's edition is based on the coordinate plane, so there is real confusion about whether a proof is supposed to be in the Euclidean plane (no coordinates) or in the coordinate plane.

A strong point of this chapter is that after a rather lengthy review of the various forms of the equation of a line, there are some substantial applications of the algebra to constructing parallel lines and perpendicular bisectors, finding distance from a point to a line, and other applications. On the other hand, the development of angles defined by transversals introduces an unusually large number of terms for the pairs of angles; the attention necessary for mastering this terminology diverts the narrative from more important geometric content. The chapter also provides rather weak support for understanding and proving, as opposed to memorizing, these properties. It is unfortunate that the slope properties of parallels and perpendiculars are presented as postulates rather than as theorems than can be explained and proved (with algebra and at least informally with geometry). There are some exercises that call for proof, but there is little support for learning how to write proofs. And the logical flaws in the development of the parallel postulate and in the treatment of distance pointed out above detract significantly from the mathematical rigor and clarity.

Chapter 6 deals with parallelograms and polygons. Section 6.1 presents the interior and exterior angle sum formulas for a convex polygon. These formulas are considered in a number of exercises about general polygons and also previews of some special cases. In Section 6.2 the standard properties of parallelograms are stated and proved (i.e., one example of a proof, the rest as exercises). Some examples of parallelogram arms from the real world are shown. In Section 6.3 sufficient conditions for a quadrilateral to be a parallelogram are proved. Section 6.4 is about rectangles, with a proof of equal diagonals being a necessary and sufficient condition for a parallelogram to be a rectangle. Section 6.5 is about rhombi and squares, including the definitions and properties of the diagonals. This chapter develops the ideas clearly and correctly, with several examples of proofs provided as models. The inclusion of examples for the coordinate plane meets the requirements of Performance Expectation G.4.C.

The *Glencoe* text covers the topics required by the Washington Standards. In many places the treatment is clear and correct. But as noted in the section summaries, there are several instances of logical flaws, a conflation of genuine postulates and unproved theorems and some confusing mathematical statements that detract from the text.

4.2.4 Prentice-Hall Geometry

Chapter 1 lays significant groundwork for the study of geometry. Topics include informal geometry, important definitions (e.g., parallel and skew lines, parallel planes, perpendicular lines), compass and straightedge constructions, the coordinate plane (e.g., formula for the midpoint of the segment), and the distance formula (based on the Pythagorean Theorem). The text carefully distinguishes the use of the word "segment" from the word "line." Some exercises contrast circular definitions with the use of undefined terms in mathematics, and the discussion addresses the tension between the logical development of geometry as an axiomatic system and the fact that students will have already studied informal geometry in earlier grades. It attempts to make clear what is proved and what is not yet proved.

The development of logical tools for proof is taken up systematically in Chapter 2.

Section 2.1 introduces conditional (if-then) statements right away, with many examples, including rewording of statements not in if-then form into if-then form. Counterexamples and converses (and the truth value of the converse) are introduced and illustrated. The chapter also includes Venn diagrams and standard arrow symbols. Section 2.2 contains a careful introduction to biconditional statements and definitions. Section 2.3 is about deduction, including the Law of Detachment and the Law of Syllogism. Examples and problems focus on the effective and correct use of these tools. Section 2.5 centers on the use of equations and algebra for solving questions in geometry. Section 2.6 uses these algebraic tools to make angle computations, including proving that vertical angles are equal. The chapter does a good job of presenting the important tools of logic and proof and addressing possible points of confusion. It is efficient in that it does not digress into a study of logic or algebra beyond what is needed for geometry.

Chapter 3 addresses parallel/perpendicular lines. Section 3.1 defines three pairs of angles formed by a transversal of any pair of lines and then moves to the case of parallel lines with the postulate that corresponding angles formed by a transversal intersecting a pair of parallel lines are congruent. The other angle relations formed by parallels and a transversal are proved. The teacher notes correctly point out that the Corresponding Angle Postulate is a variation of the Euclidean Parallel Postulate. This section is distinguished in that it moves briskly from definitions to the geometrical content of angles and parallels. Section 3.2 contains a postulate and then theorems stating the usual conditions that congruence of one pair of angles (corresponding, or alternate interior, etc.) formed by a transversal and two lines implies that the two lines are parallel. The theorems are correctly labeled as converses of the theorems in the previous section. Section 3.3 is about parallel and perpendicular lines. Perpendicular transversals are used to give a correct proof that two lines parallel to the same line are parallel. Section 3.4 proves that the sum of the angles of a triangle is 180 degrees. By proving this theorem in the chapter on parallels, the text provides an interesting and powerful application of the theory of angles and parallels. After this theorem, the exterior angle theorem is proved and classifications of triangles by angle are introduced. Section 3.5 proves angle sum theorems (both interior and exterior) for convex polygons. Sections 3.6 and 3.7 deal with the slopes of parallel and perpendicular lines. These relations are correctly presented as concepts that will be proved later rather than as postulates. Section 3.8 presents step-by2008 step straightedge and compass constructions of parallel and perpendicular lines. The treatment of parallels in this chapter presents the theorems about angles and parallels concisely but effectively. Distance does not appear in the section (thus avoiding some logical sequence problems). The mathematics is correct, including the appropriate distinction between logically necessary postulates and facts that are really theorems than can be proved later. Also, the understanding of the parallel postulate is correct.

Chapter 6 is about quadrilaterals, including application of the angle sum theorem for convex polygons, which was proved in Chapter 3. Section 6.1 begins with the definitions of special quadrilaterals, along with a diagram relating the logical relationships among the various kinds of quadrilaterals. Exercises develop examples and consequences of the definitions, including examples in the coordinate plane. Section 6.2 presents the standard properties of parallelograms. The equality of opposite sides is proved in a detailed proof. Included is one useful theorem that is often not stated: if three parallel lines cut off two congruent segments on one transversal, then they cut off two congruent segments on any transversal (a situation that occurs multiple times with notebook paper or street grids). Section 6.3 contains the sufficient conditions to prove that a quadrilateral is a parallelogram. Careful proofs are given of two of the theorems. Examples and investigations are included. The topic of Section 6.4 is special parallelograms, namely rhombuses and rectangles. Theorems about the diagonals are proved (i.e., necessary and sufficient conditions). Numerous exercises are included, some about problem solving and some asking for proofs. This development of the theory of parallelograms is complete and clear. The extra theorem about transversals and congruent segments is an interesting and useful application. The examples of proofs do a good job of making clear how proofs are written.

The selected topics from the Washington Standards are covered fully in *Prentice-Hall Geometry*. Some things that distinguish this text are the unusual placement of the angle sum theorems and the inclusion of an additional theorem about parallels. More importantly, the text shows good mathematical judgment. The relationship between postulates about parallels and angles and the Euclidean parallel postulate is understood correctly. The text refrains from labeling every unproved fact as a postulate, instead stating them as “principles” that are merely as-yet unproved theorems. Also, the text avoids some tricky points making hidden and unproved assumptions about distance and parallelism. There is a generous supply of exercises and activities.

4.2.5 Conclusions: Geometry

The *Mathematics Standards* state that students should know, prove, and apply theorems about angles that arise from parallel lines intersected by a transversal. The development adopted by the reviewed texts is to assume as a postulate that for any two parallel lines intersected by a transversal corresponding angles are congruent. It is immediate to prove that a number of pairs of angles are either congruent or supplementary (for example, alternating interior angles are congruent). Then, as a second postulate, the converse of the first postulate is assumed. After this, it is proved that the necessary conditions in the earlier theorems are in fact sufficient conditions.

One important “backstory” for this development is that these postulates imply the Euclidean Parallel Postulate (EPP). To be precise, the second postulate can be proved as a theorem in Euclidean geometry and the first postulate is equivalent to the EPP. Some of the textbooks try to include some of this background, more or less successfully as the reviews note. It is not strictly necessary for students to know this background for their study of geometry, but if the choice is to introduce the EPP, it would be better to tell the story correctly.

The texts differ in the accuracy and completeness with which they present the relevant mathematics. *Holt Geometry* and *Prentice-Hall Geometry* seem to be the most successful in this regard. Teachers might have to be more careful in explicating the mathematics of the other two texts.

4.3 Integrated Mathematics

All of the integrated mathematics materials were three-book series. The same threads were examined here as were examined in the Algebra 1/Algebra 2 and Geometry materials.

One characteristic that distinguishes integrated mathematics materials from more traditional materials is the extensive use of contexts and applications as the focus of attention. Mathematics ideas are typically not presented as “naked” mathematics, but rather as ways to solve problems. This does not mean that the mathematics is less important or less well developed, but it does make a review of mathematical soundness somewhat more complex.

4.3.1 Core-Plus Mathematics

Functions. In Course 1, quadratic functions (Unit 7) are introduced through specific examples (e.g., projectile paths). This specific approach has the potential to create “stereotypical images” in students’ minds that may be difficult to overcome to create a general understanding of quadratic functions. It appears, however, that by the time students work through Investigation 3 a general understanding should have developed. The teacher's role in debriefing students' work is probably critical so that students understand how the parameters for the general quadratic function influence the shape and position of the graph.

In Course 2, quadratic functions are treated as one kind of nonlinear function (Unit 5). This is a strength mathematically, since it helps reinforce the similarities and difference among different kinds of nonlinear functions. It is in this unit that domain and range are emphasized (Lesson 1, Investigation 2) and factoring is developed (Investigation 3). The area model (i.e., algebra tiles) is used to motivate techniques for factoring. Solving of quadratic equations is developed, and the quadratic formula is presented, but it appears to be developed only in the "On Your Own" section of problems/exercises. Lesson 2 focuses on Nonlinear Systems; this provides an immediate application of what was dealt with in Lesson 1.

In Course 3, quadratic functions reappear in Unit 5, Lesson 2: Quadratic Polynomials. Completing the square is the focus of Investigation 1; by this point, all students should be intellectually prepared to understand the mathematics of this idea at a deep level. The vertex form of the equation is addressed here, and complex numbers are introduced with the obvious extension to quadratic equations with no real solutions can be examined.

Geometry. In Course 1, the study of properties of figures begins in Unit 6. "The focus here is on careful visual reasoning, not on formal proof." (Formal proof is addressed extensively in Course 3.) Unit 6 is "developed and sequenced in a manner consistent with the van Hiele levels of geometric thinking." Senk's data (1986) suggest strongly that students who attempt to study proof before the development of Level 2 thinking (e.g., Fuys, Geddes, & Tischler, 1988; Van Hiele, 1986) are unlikely to be successful. Unit 6 is organized to help students develop Level 2 thinking. Because the study of formal proof is delayed another year, there are additional opportunities for this kind of thinking to develop.

Unit 6, Lesson 1, deals with a variety of topics at an informal level, including conditions that determine triangles or quadrilaterals (e.g., triangle inequality), angle sums for polygons, SSS/SAS/ASA properties of triangles, reasoning about shapes, and the Pythagorean Theorem. Some constructions are included as an extension of this work. Lesson 2 addresses symmetries of figures, angle sums of polygons, and tessellations. The tasks here emphasize relationships among different shapes; these help students internalize Level 2 van Hiele thinking. Specific attention is paid to interior and exterior angles of polygons. Lesson 3 deals with three-dimensional shapes. This work, too, is informal. It is much more exploratory, since students are likely to have less well-developed understanding of three-dimensional shapes.

The primary attention to geometry in Course 2 is coordinate geometry. This is important but does not relate directly to the threads being reviewed here.

In Course 3, Unit 1 addresses proof. The unit begins with an introduction to logical reasoning set in many different contexts, not just geometry. This is an obvious strength for the study of proof. Lessons 2, 3, and 4 address proof in geometry (mainly study of angles when parallel lines are cut by a transversal), algebra, and statistics. Both in this Unit and in Unit 3, the teacher notes are extensive, with considerable detail provided for each of the proofs. These notes would support teachers well in leading discussions that were effective at helping students internalize the critical mathematics ideas.

Unit 3 addressed triangle similarity (Lesson 1) and congruence (Lesson 2). In Lesson 1, students explore a variety of conjectures, for example, all isosceles right triangles are similar. There are numerous applications of similarity which provide a rationale and motivation for proofs. As one would expect in a "proof unit," there are numerous classic mathematics relationships established and proved. In Lesson 2 congruence is studied as a special case of similarity. Included are the classic triangle congruence theorems, with attention also paid to perpendicular bisectors of sides, angle bisectors, and medians. This is followed by an equally extensive study of the properties of quadrilaterals, with particular attention to parallelograms.

In summary, the mathematics in *Core Plus* is mathematically sound and very well sequenced to support student learning at a deep level.

4.3.2 SIMMS Integrated Mathematics

Functions. In Level 1, quadratic functions are addressed in Module 10. Distance/time graphs are used as a context to support comparison of these graphs to determine average velocity over a time interval, leading to linear modeling for objects moving at constant speed. Quadratic functions are introduced in Activity 3; topics include coordinates of the vertex, vertex form of quadratic function rule, families of functions (based on $y = x^2$), and translation of parabola graphs. The Chapter ends with an exploration of the quadratic modeling of data.

One difficulty in analyzing the Teacher's Guide is that there is very little discussion of the mathematics; detailed answers are provided for each task, but there is no rationale provided for the sequencing of these tasks. It might be difficult for some teachers to lead appropriate debriefing of the exercises so that students truly internalize mathematical understanding. Merely solving the tasks correctly does guarantee depth of understanding.

In Level 2, quadratics are addressed in Module 6 as part of the study of polynomials, with parabolas highlighted in Activity 2. Topics addressed include fitting a parabola to three non-collinear points, roots and factors of polynomials, and effects of changing the parameter, *a*, in the general form of a quadratic function. Embedding quadratic functions in a more general context is a strength for supporting students' understanding.

In Level 3, Module 11, transformations of functions are addressed. This is a general treatment, though some examples are quadratic functions. There does not appear to be a significant development of quadratic functions, *per se*, in Level 3.

Geometry. In Level 1, Module 1, simple ideas about angles are used to introduce techniques for studying mathematics. There is little development here. The Activities in Module 4 address surface area of three-dimensional figures, tessellations, and area of regular polygons. These ideas "feel" disconnected, with little obvious attempt to highlight common features of the ideas.

In Level 2, Modules 3 and 7 each address geometric ideas, but again the connections among them are not immediately obvious. Module 3 addresses area of regular polygons and surface area and volume of three-dimensional shapes. Module 7 addresses angles formed by a transversal of parallel lines, tangents and secants to circles, and dilations. Many teachers might need help in communicating to students what key mathematics ideas underlie the tasks. Module 12 is a more traditional treatment of proof. Three areas are addressed: Pythagorean Theorem, triangles, and quadrilaterals. However, there may not be enough tasks to support deep understanding by students of the nature of proof.

In Level 3, Module 6 is a more general treatment of proof. It is strange that this Module is after the Module in Level 2 on proof of triangle and quadrilateral theorems. Certainly students by Level 3 should be ready to learn this material, but it might also have been useful prior to the work with congruent triangles in Level 2.

In summary, the development of mathematical ideas is difficult to follow in *SIMMS*. This observation seems reinforced by examination of the alignment grid provided by the publisher. Many of the Performance Expectations are addressed in parts of problems scattered across a wide range of pages. It seems likely that some teachers might have difficult in helping students internalize the mathematical ideas based on the tasks they have completed. Also, the Modules seem too short to support in-depth development of mathematical ideas.

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